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TWO LINNIK-TYPE PROBLEMS FOR AUTOMORPHIC L -FUNCTIONS

JIANYA LIU, YAN QU & JIE WU

ABSTRACT

Let $m \geq 2$ be an integer, and π an irreducible unitary cuspidal representation for $GL_m(\mathbb{A}_{\mathbb{Q}})$, whose attached automorphic L -function is denoted by $L(s, \pi)$. Let $\{\lambda_{\pi}(n)\}_{n=1}^{\infty}$ be the sequence of coefficients in the Dirichlet series expression of $L(s, \pi)$ in the half-plane $\Re s > 1$. It is proved in this paper that, if π is such that the sequence $\{\lambda_{\pi}(n)\}_{n=1}^{\infty}$ is real, then the first sign change in the sequence $\{\lambda_{\pi}(n)\}_{n=1}^{\infty}$ occurs at some $n \ll Q_{\pi}^{1+\varepsilon}$, where Q_{π} is the conductor of π , and the implied constant depends only on m and ε . This improves the previous bound with the above exponent $1 + \varepsilon$ replaced by $m/2 + \varepsilon$. A result of the same quality is also established for $\{\Lambda(n)a_{\pi}(n)\}_{n=1}^{\infty}$, the sequence of coefficients in the Dirichlet series expression of $-\frac{L'}{L}(s, \pi)$ in the half-plane $\Re s > 1$.

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1. INTRODUCTION

Let π be a cuspidal automorphic representations of $GL_m(\mathbb{A}_{\mathbb{Q}})$. We can attach to any such π an automorphic L -function $L(s, \pi)$, which is defined by an Euler product, and for $\sigma = \Re s > 1$, it can be represented by an absolutely convergent Dirichlet series

$$L(s, \pi) = \sum_{n=1}^{\infty} \frac{\lambda_{\pi}(n)}{n^s}. \quad (1.1)$$

The sequence $\{\lambda_{\pi}(n)\}_{n=1}^{\infty}$ consists of complex numbers, which we always normalize so that $\lambda_{\pi}(1) = 1$. It may happen that $\lambda_{\pi}(n)$ is real for all $n \geq 1$; for example, it is the case when π is a self-contragredient representation for $GL_m(\mathbb{A}_{\mathbb{Q}})$ with trivial central character.

The purpose of this note is to continue the study in [4] of Linnik-type problems for automorphic L -functions $L(s, \pi)$. The reader is referred to [4] for general philosophy, history, and results in this direction.

The first Linnik-type problem we are going to study is, in the case $\lambda_\pi(n)$ is real for all $n \geq 1$, to find the first n such that $\lambda_\pi(n) < 0$. Our result is as follows.

Theorem 1.1. *Let $m \geq 2$ be an integer and let π be an irreducible unitary cuspidal representation for $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$. If $\lambda_\pi(n)$ is real for all $n \geq 1$, then there is some n satisfying*

$$n \ll_{m,\varepsilon} Q_\pi^{1+\varepsilon} \quad (1.2)$$

such that $\lambda_\pi(n) < 0$. The constant implied in (1.2) depends only on m and ε . In particular, the result is true for any self-contragredient irreducible unitary cuspidal representation π for $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$ with trivial central character.

The second Linnik-type problem considered in this note concerns sign changes in the sequence $\{\Lambda(n)a_\pi(n)\}_{n=1}^\infty$, which appears naturally in the Dirichlet series expression of the logarithmic derivative of $L(s, \pi)$ in the half-plane $\sigma > 1$:

$$\frac{d}{ds} \log L(s, \pi) = - \sum_{n=1}^{\infty} \frac{\Lambda(n)a_\pi(n)}{n^s}. \quad (1.3)$$

Here $\Lambda(n)$ is the von Mangoldt function, and $\{\Lambda(n)a_\pi(n)\}_{n=1}^\infty$ is a sequence of complex numbers. The following is a Linnik-type theorem for this sequence.

Theorem 1.2. *Let $m \geq 2$ be an integer and let π be an irreducible unitary cuspidal representation for $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$. If $\Lambda(n)a_\pi(n)$ is real for all $n \geq 1$, then there is some n satisfying*

$$n \ll_{m,\varepsilon} Q_\pi^{1+\varepsilon} \quad (1.4)$$

such that $\Lambda(n)a_\pi(n) < 0$. The constant implied in (1.4) depends only on m and ε . In particular, the result is true for any self-contragredient irreducible unitary cuspidal representation π for $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$ with trivial central character.

Theorems 1.1 and 1.2 improve significantly the corresponding results in [4], where the exponents $1 + \varepsilon$ in (1.2) and (1.4) were replaced by the weaker exponents $m/2 + \varepsilon$. Note that our present exponents in (1.2) and (1.4) are independent of the dimension m of $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$. New ideas leading to these improvements will be explained before Lemma 3.1 and before Lemma 3.3.

2. AUTOMORPHIC L -FUNCTIONS

Let π be an irreducible unitary cuspidal representation $\pi = \otimes_p \pi_p$ of $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$. To every prime p at which π_p is unramified, there is an associated set of m nonzero complex Satake parameters

$\{\alpha_\pi(p, j)\}_{j=1}^m$, out of which one may define local L -functions

$$L(s, \pi_p) = \prod_{j=1}^m \left(1 - \frac{\alpha_\pi(p, j)}{p^s}\right)^{-1}. \quad (2.1)$$

At p where π_p is ramified, the local L -function is defined in terms of the Langlands parameters of π_p . It is possible to write the local factors at ramified primes in the form of (2.1) with the convention that some of the $\alpha_\pi(p, j)$'s may be zero. Here it is appropriate to point out that the coefficients $\{\lambda_\pi(n)\}_{n=1}^\infty$ and $\{a_\pi(p^k)\}_{k=1}^\infty$ in (1.1) and (1.3) are actually defined, respectively, by

$$\lambda_\pi(n) = \prod_{p^\nu \parallel n} \left\{ \sum_{\nu_1 + \dots + \nu_m = \nu} \alpha_\pi(p, 1)^{\nu_1} \dots \alpha_\pi(p, m)^{\nu_m} \right\}, \quad (2.2)$$

and

$$a_\pi(p^k) = \sum_{j=1}^m \alpha_\pi(p, j)^k. \quad (2.3)$$

At the archimedean place ∞ , a set of m complex Langlands parameters $\{\mu_\pi(j)\}_{j=1}^m$ is associated to π_∞ . The local factor at ∞ is defined to be

$$L(s, \pi_\infty) = \pi^{-\frac{ms}{2}} \prod_{j=1}^m \Gamma\left(\frac{s + \mu_\pi(j)}{2}\right). \quad (2.4)$$

For the parameters $\{\alpha_\pi(p, j)\}_{j=1}^m$ and $\{\mu_\pi(j)\}_{j=1}^m$, trivial bounds state that

$$|\alpha_\pi(p, j)| \leq \sqrt{p}, \quad |\Re \mu_\pi(j)| \leq \frac{1}{2}.$$

The finite-part L -function $L(s, \pi)$ is defined by products of local factors

$$L(s, \pi) = \prod_{p < \infty} L(s, \pi_p), \quad (2.5)$$

and the complete L -function $\Phi(s, \pi)$ is defined by

$$\Phi(s, \pi) = L(s, \pi_\infty) L(s, \pi). \quad (2.6)$$

This complete L -function extends to an entire function on the whole complex plane via its functional equation

$$\Phi(s, \pi) = \varepsilon_\pi N_\pi^{\frac{1}{2}-s} \Phi(1-s, \tilde{\pi}) \quad (2.7)$$

where $\tilde{\pi}$ is the contragredient of π , ε_π a complex number of modulus 1, and N_π a positive integer called the arithmetic conductor of π . Finally $\Phi(s, \pi)$ is of order one and bounded in the vertical strips. The reader is referred to e.g. Cogdell [1] for proofs of these properties.

The functional equation (2.7) can be re-written as

$$L(s, \pi) = \varepsilon_\pi N_\pi^{\frac{1}{2}-s} G(s) L(1-s, \tilde{\pi}), \quad (2.8)$$

where

$$G(s) = \frac{L(1-s, \tilde{\pi}_\infty)}{L(s, \pi_\infty)}. \quad (2.9)$$

The following lemma gives an estimate for $G(s)$ on the vertical line $\sigma = -H$, avoiding the poles of the nominator of $G(s)$. Its proof is a simple application of Stirling's for the Γ -function, so we omit the details.

Lemma 2.1. *For each positive integer N , there is an $H \in [N, N+1]$, such that on the line $\sigma = -H$ we have*

$$G(-H+it) \ll_{H,m} (1+|t|)^{m(\frac{1}{2}+H)} \prod_{j=1}^m (1+|\mu_\pi(j)|)^{\frac{1}{2}+H}. \quad (2.10)$$

Following Iwaniec-Sarnak [3], we define the analytic conductor of $L(s, \pi)$ as

$$Q_\pi(t) = N_\pi \prod_{j=1}^m (1+|it+\mu_\pi(j)|). \quad (2.11)$$

Setting $t = 0$ in the above definitions, we write

$$Q_\pi = Q_\pi(0) \quad (2.12)$$

which is called the conductor of π .

3. PREPARATIONS FOR THEOREM 1.1

The purpose of this section is to establish the preliminaries required by Theorems 1.1. Suppose that

$$\lambda_\pi(n) \geq 0 \quad \text{for all } n \leq x. \quad (3.1)$$

We start with the sum

$$S_\pi(x) = \sum_n \lambda_\pi(n) w\left(\frac{n}{x}\right), \quad (3.2)$$

where $w(x)$ is a non-negative real valued function of C_c^∞ with compact support in $[0, 1]$. In this note we specify

$$w(x) := \begin{cases} \exp(-x^{-\frac{1}{4m}}) \exp\{- (1-x)^{-\frac{1}{4m}}\} & \text{if } x \in (0, 1), \\ 0 & \text{otherwise.} \end{cases} \quad (3.3)$$

We have good reasons to choose this specific weight function; the reader will see, in the discussion after the proof of Theorem 1.1 in §4, that the better-looking function

$$\begin{cases} \exp(-x^{-1}) \exp\{-(1-x)^{-1}\} & \text{if } x \in (0, 1) \\ 0 & \text{otherwise} \end{cases} \quad (3.4)$$

does not do the job.

Theorem 1.1 will follow from upper and lower bound estimates for $S_\pi(x)$. Our present strategy of establishing upper and lower bound estimates for $S_\pi(x)$ is different from that in [4]. In getting our new upper bound, we apply Landau's method instead of the direct application of convexity bound of $L(s, \pi)$ as in [4], and this results in extra savings. Our new upper bound for $S_\pi(x)$ is as follows.

Lemma 3.1. *For each positive integer N , there is an $H \in [N, N+1]$, such that*

$$S_\pi(x) \ll_{H,m} x^{-H} Q_\pi^{\frac{1}{2}+H}, \quad (3.5)$$

where the implied constant depends on H and m .

Proof. We note that, for any positive integer k , the derivative $w^{(k)}(x)$ has exponential decay as $x \rightarrow 0+$ or $1-$. Consequently, the Mellin transform

$$W(s) = \int_0^\infty w(x) x^{s-1} dx$$

is an analytic function of s . By repeated partial integration, we have

$$|W(\sigma + it)| \ll_{A,\sigma,m} \frac{1}{(1 + |t|)^A} \quad (3.6)$$

for arbitrary positive constant A , and in particular for $\sigma \leq 2$. Now we apply Mellin inversion, to get

$$w(x) = \frac{1}{2\pi i} \int_{(2)} W(s) x^{-s} ds,$$

where (c) means the vertical line $\sigma = c$. Inserting this back to (3.2), and then using Dirichlet series expansion (1.1), we have

$$\begin{aligned} S_\pi(x) &= \frac{1}{2\pi i} \sum_n \lambda_\pi(n) \int_{(2)} W(s) \left(\frac{n}{x}\right)^{-s} ds \\ &= \frac{1}{2\pi i} \int_{(2)} x^s W(s) L(s, \pi) ds, \end{aligned}$$

where the interchange of summation and integral is guaranteed by the absolute convergence of (1.1) on the line $\sigma = 2$. A pre-convexity bound like

$$L(1/2 + it, \pi) \ll Q_\pi(t)^B, \quad (3.7)$$

where $B > 0$ is some constant, can be obtained by standard analytic method via the functional equation (2.7). Actually it has been proved by Harcos [2] that any constant $B > \frac{1}{4}$ is acceptable in (3.7). It follows that

$$L(\sigma + it, \pi) \ll_\varepsilon Q_\pi(t)^{\mu(\sigma) + \varepsilon} \quad (3.8)$$

with

$$\mu(\sigma) = \begin{cases} (1 - \sigma)/2 & \text{if } 0 \leq \sigma \leq 1, \\ 1 - \sigma/2 & \text{if } \sigma \leq 0. \end{cases}$$

Since both $W(s)$ and $L(s, \pi)$ are entire, we may apply (3.6) and (3.8) to shift the contour above to the vertical line $\sigma = -H$, getting

$$S_\pi(x) = \frac{1}{2\pi i} \int_{(-H)} x^s W(s) L(s, \pi) ds, \quad (3.9)$$

where $H \in [N, N + 1]$ is a real number decided by Lemma 2.1.

Now we apply a classical idea of Landau to insert the functional equation (2.8) into (3.9), getting

$$\begin{aligned} S_\pi(x) &= \frac{1}{2\pi i} \int_{(-H)} x^s W(s) \varepsilon_\pi N_\pi^{\frac{1}{2}-s} G(s) L(1-s, \tilde{\pi}) ds \\ &= \frac{1}{2\pi i} \int_{(-H)} x^s W(s) \varepsilon_\pi N_\pi^{\frac{1}{2}-s} G(s) \left(\sum_{n=1}^{\infty} \frac{\lambda_{\tilde{\pi}}(n)}{n^{1-s}} \right) ds \\ &= \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{\overline{\lambda_\pi(n)}}{n^{1+H}} \int_{(-H)} x^s W(s) \varepsilon_\pi N_\pi^{\frac{1}{2}-s} G(s) n^{s+H} ds. \end{aligned}$$

Here the interchange of summation and integral is guaranteed by the absolute convergence of the Dirichlet series as well as the rapid decay of $W(s)$ in (3.6). Using these facts again, and

inserting (2.10) into the last integral, we get

$$\begin{aligned}
S_\pi(x) &\ll_{H,m} \int_{(-H)} \left| x^s W(s) \varepsilon_\pi N_\pi^{\frac{1}{2}-s} G(s) ds \right| \\
&\ll_{H,m} x^{-H} N_\pi^{\frac{1}{2}+H} \prod_{j=1}^m (1 + |\mu_\pi(j)|)^{\frac{1}{2}+H} \\
&= x^{-H} Q_\pi^{\frac{1}{2}+H}.
\end{aligned}$$

This proved Lemma 3.1. \square

To get a suitable lower bound for $S_\pi(x)$, we will need the following result, which is Lemma 5.3 in [4].

Lemma 3.2. *Let $m \geq 2$ be an integer and let π be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_\mathbb{Q})$. For any prime p such that π_p is unramified, we have*

$$|\lambda_\pi(p^m)| + |\lambda_\pi(p^{m-1})| + \cdots + |\lambda_\pi(p)| \geq \frac{1}{m}.$$

In [4], the above lemma was applied to establish the lower bound

$$\begin{aligned}
\sum_{n \leq x} \lambda_\pi(n) &\geq \sum_{\substack{n \leq x \\ (n, N_\pi) = 1}} \lambda_\pi(n) \\
&\geq \sum_{\substack{p \leq x^{1/m} \\ p \nmid N_\pi}} \{ \lambda_\pi(p^m) + \cdots + \lambda_\pi(p) \} \\
&\gg_m \sum_{\substack{p \leq x^{1/m} \\ p \nmid N_\pi}} 1 \gg_m \frac{x^{1/m}}{\log x} - \log N_\pi,
\end{aligned} \tag{3.10}$$

which is enough to derive the main theorem there. However, (3.10) does not imply a useful lower for $S_\pi(x)$, since the sum $S_\pi(x)$ actually counts the contribution essentially from $n \in [\rho x, (1-\rho)x]$ for some small $\rho > 0$, which follows from the fact that our weight function $w(x)$ has exponential decay when $x \rightarrow 0+$ or $1-$. To get around this difficulty, we must find at least one integer n_0 , which is close to neither 0 or 1, such that $\lambda_\pi(n_0)$ relatively large. This is achieved in the following lemma.

Lemma 3.3. (i) *There is an integer n_0 with canonical decomposition*

$$n_0 = p_0^{\nu_0} p_1^{\nu_1} \cdots p_r^{\nu_r}, \tag{3.11}$$

where p_j 's are distinct primes not dividing N_π and ν_j 's are positive integers not exceeding m such that

$$\lambda_\pi(p_j^{\nu_j}) \geq \frac{1}{m^2} \quad \text{for all } j = 0, 1, \dots, r. \quad (3.12)$$

In addition p_0 is the smallest prime not dividing N_π .

(ii) The above n_0 falls in the interval $[\rho x, (1 - \rho)x]$, where

$$\rho = \frac{1}{p_0^m + 1}. \quad (3.13)$$

Proof. Our proof is of combinatorial nature. We begin by applying the pigeonhole principle to Lemma 3.2 to see that, for every prime $p \nmid N_\pi$, there exists a positive integer $\nu \leq m$ depending on both p and π , such that

$$\lambda_\pi(p^\nu) \geq \frac{1}{m^2}. \quad (3.14)$$

A prime power p^ν is called *good* with respect to p , if $p \nmid N_\pi$ and ν is the smallest positive integer such that (3.14) holds. An immediate consequence of the definition is that two good prime powers $p_1^{\nu_1}$ and $p_2^{\nu_2}$ are different if and only if $p_1 \neq p_2$.

Let $0 < \rho < 1$ be a parameter, not yet specified. Take the largest good prime power $< \rho x$, say $p_1^{\nu_1}$, and form the interval

$$\left[\frac{\rho x}{p_1^{\nu_1}}, \frac{(1 - \rho)x}{p_1^{\nu_1}} \right].$$

Then take the largest good prime power $< \rho x / p_1^{\nu_1}$, say $p_2^{\nu_2}$, and form the second interval

$$\left[\frac{\rho x}{p_1^{\nu_1} p_2^{\nu_2}}, \frac{(1 - \rho)x}{p_1^{\nu_1} p_2^{\nu_2}} \right].$$

We repeat the process until there is an interval

$$\left[\frac{\rho x}{p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r}}, \frac{(1 - \rho)x}{p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r}} \right] \quad (3.15)$$

with all these good prime powers satisfying

$$p_j^{\nu_j} < \frac{\rho x}{p_1^{\nu_1} p_2^{\nu_2} \cdots p_{j-1}^{\nu_{j-1}}} \quad \text{for all } j = 1, \dots, r, \quad (3.16)$$

such that there is no good prime power smaller than $\rho x / (p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r})$. In particular, for the good prime power $p_0^{\nu_0}$ where p_0 is the smallest prime not dividing N_π , we must have

$$p_0^{\nu_0} \geq \frac{\rho x}{p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r}}. \quad (3.17)$$

We want to show that $p_0^{\nu_0}$ lies in the interval (3.15) for some suitably chosen ρ . To this end, specify ρ as in (3.13), so that $p_0^{\nu_0} \rho \leq 1 - \rho$. Also, (3.16) with $j = r$ states that

$$1 < \frac{\rho x}{p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r}}.$$

and hence

$$p_0^{\nu_0} < p_0^{\nu_0} \frac{\rho x}{p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r}} \leq \frac{(1 - \rho)x}{p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r}}. \quad (3.18)$$

It follows from (3.17) and (3.18) that $p_0^{\nu_0}$ indeed lies in the interval (3.15) with ρ specified as in (3.13).

Finally we write $n_0 = p_0^{\nu_0} p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r}$, and recall that all the components $p_0^{\nu_0}, \dots, p_r^{\nu_r}$ above are good prime powers which are mutually different. It follows by construction that all the primes p_0, \dots, p_r are different and satisfy $p_j \nmid N_\pi$. Thus we have (3.11) and (3.12). Since $p_0^{\nu_0}$ lies in the interval (3.15), we have $n_0 \in [\rho x, (1 - \rho)x]$. This proves the lemma. \square

Lemma 3.4. *We have the lower bound*

$$S_\pi(x) \gg_m x^{-(2/\log 2) \log m} \exp(-c_0 \sqrt{\log N_\pi}), \quad (3.19)$$

where $c_0 > 0$ is an absolute constant and the implied constant depends on m .

Proof. We start from the trivial observation that

$$S_\pi(x) \geq \lambda_\pi(n_0) w\left(\frac{n_0}{x}\right), \quad (3.20)$$

where n_0 is decided by Lemma 3.3. The assertion of the current lemma will follow from lower bounds for $\lambda_\pi(n_0)$ and for $w(n_0/x)$.

To get a lower bounds for $\lambda_\pi(n_0)$, we recall that

$$\lambda_\pi(n_1 n_2) = \lambda_\pi(n_1) \lambda_\pi(n_2) \quad \text{for } (n_1, n_2) = 1, (n_1 n_2, N_\pi) = 1.$$

This together with (3.11) and (3.12) implies that

$$\lambda_\pi(n_0) = \lambda_\pi(p_0^{\nu_0} p_1^{\nu_1} \cdots p_r^{\nu_r}) = \lambda_\pi(p_0^{\nu_0}) \lambda_\pi(p_1^{\nu_1}) \cdots \lambda_\pi(p_r^{\nu_r}) \geq m^{-2(r+1)}.$$

An elementary argument gives

$$r + 1 \leq \frac{\log n_0}{\log 2} \leq \frac{\log x}{\log 2},$$

and consequently

$$\lambda_\pi(n_0) \geq m^{-\frac{2\log x}{\log 2}} > x^{-3\log m}. \quad (3.21)$$

This is the desired lower bound for $\lambda_\pi(n_0)$.

To get a useful lower bound for $w(n_0/x)$, we must analyze the weight function $w(x)$ carefully. It is easy to check that, for $x \in (0, 1)$,

$$w'(x) = \frac{1}{4m} w(x) \{x^{-\frac{1}{4m}-1} - (1-x)^{-\frac{1}{4m}-1}\},$$

and consequently $w(x)$ is increasing and decreasing, respectively, in the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. Lemma 3.3 asserts that $n_0/x \in [\rho, 1-\rho]$, and hence

$$w\left(\frac{n_0}{x}\right) \geq w(\rho) = \exp\left(-\rho^{-\frac{1}{4m}}\right) \exp\left\{-\left(1-\rho\right)^{-\frac{1}{4m}}\right\}. \quad (3.22)$$

From (3.13) we infer $\rho \leq \frac{1}{5}$, and hence

$$\exp\left\{-\left(1-\rho\right)^{\frac{1}{4m}}\right\} \geq \exp\left\{-\left(4/5\right)^{\frac{1}{4m}}\right\}.$$

To evaluate the other exponential factor in (3.22), we apply (3.13) again to deduce that $\rho \geq (2p_0)^{-m}$, and consequently

$$\exp\left(-\rho^{-\frac{1}{4m}}\right) \geq \exp\left\{-\left(2p_0\right)^{\frac{1}{4}}\right\}.$$

Now we need to know the largest possible value of p_0 , which is by definition the smallest prime not dividing N_π . The largest possible value of p_0 occurs when $N_\pi = q_1 \cdots q_s$ is the product of the first s primes, and in this case $p_0 = q_{s+1}$, the $(s+1)$ -th prime number. Since

$$\log N_\pi = \sum_{p \leq q_s} \log p \sim \frac{q_s}{\log q_s},$$

we must have $q_s \leq c_1(\log N_\pi) \log \log N_\pi \leq c_1(\log N_\pi)^2$ with an absolute constant $c_1 > 0$. It follows that

$$p_0 = q_{s+1} \leq 2q_s \leq 2c_1(\log N_\pi)^2,$$

which is the largest possible value of p_0 . It follows that

$$\exp\left(-\rho^{-\frac{1}{4m}}\right) \geq \exp\left\{-\left(2p_0\right)^{\frac{1}{4}}\right\} \geq \exp\left\{-\left(4c_1\right)^{\frac{1}{4}}\sqrt{\log N_\pi}\right\},$$

which is the desired lower bound for the first exponential factor in (3.22).

Inserting everything back to (3.22), we get

$$w\left(\frac{n_0}{x}\right) \gg_m \exp\left\{-\left(4c_1\right)^{\frac{1}{4}}\sqrt{\log N_\pi}\right\},$$

which in combination with (3.20) and (3.21) gives the assertion of the lemma with $c_0 = (4c_1)^{\frac{1}{4}}$. \square

4. SOLUTIONS TO LINNIK-TYPE PROBLEMS

With the preparations in §3, we can now establish Theorem 1.1.

Proof of Theorem 1.1. It follows from Lemmas 3.1 and 3.4 that

$$x^{-3\log m} \exp(-c_0 \sqrt{\log N_\pi}) \ll_m S_\pi(x) \ll_{H,m} x^{-H} Q_\pi^{\frac{1}{2}+H}.$$

One therefore has

$$x < c_2 \cdot Q_\pi^E \exp(c_0 \sqrt{\log N_\pi}) \quad \text{with} \quad E = \frac{H + 1/2}{H - 3\log m},$$

where c_2 is a constant depending on H and m . Taking H sufficiently large, this becomes $x < c_3 \cdot Q_\pi^{1+\varepsilon}$, and now the constant c_3 depends on ε and m . The assertion of Theorem 1.1 is proved. \square

Remark. Had we chosen the better-looking weight function in (3.4), we would have the same Lemma 3.1, but have Lemma 3.4 replaced by

$$S_\pi(x) \gg_m x^{-3\log m} \exp\{-c_4(\log N_\pi \log \log N_\pi)^m\}$$

for some constant $c_4 > 0$. This lower bound is too weak for the above proof of Theorem 1.1 to work.

Proof of Theorem 1.2. Theorem 1.2 follows from Theorem 1.1 in the same way as done in [4], and details are therefore omitted. \square

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